accurately models the circuit in Fig. 9, where ***Q*** is electrical charge, ***L*** is inductance, ***R*** is resistance, ***C*** is capacitance and ***V(t)*** is the total voltage across the system, I hope you can see why our bungee-jumper problem has some real relevance to learning about electricity.

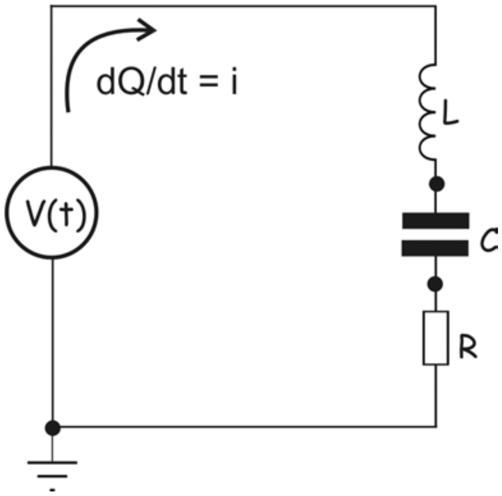


Figure 9 - An electrical circuit which is analogous with the bungee jumper

The similarity between the two equations, for the bungee-jumper and the electrical circuit, also reveals something we will return to again and again, which is that electrical properties are analogous with certain mechanical and acoustical properties. For example the electrical property of inductance is analogous with the mechanical mass. And mechanical force is analogous to acoustical pressure and electrical voltage. This enables solutions and techniques from one field to be brought to bear on another.

How to avoid differential equations - or Fourier's big cheat

With the best will in the world, it would be hard work if we had to resort to solving differential equations to determine the performance of even the simplest of electronic circuits. Happily, this isn't necessary. We already saw in the case of the bungee jumping example that, if the forcing function is a cosine function of a single frequency (**cos *ωt***), then the solution to the particular integral is an equation with a mixture of sine and cosine waves at the same frequency as the forcing function; their amplitudes depending only on the arithmetical ratios between mass, restoring force and air-resistance. Without once again going slavishly through all the maths, such a remarkable simplification may be made when dealing with sine wave (or cosine wave) excitation because,

$$\frac{d}{dt}(\cos \omega t) = -\omega \sin \omega t$$

and

$$\frac{d}{dt}(\sin \omega t) = \omega \cos \omega t$$

Similarly

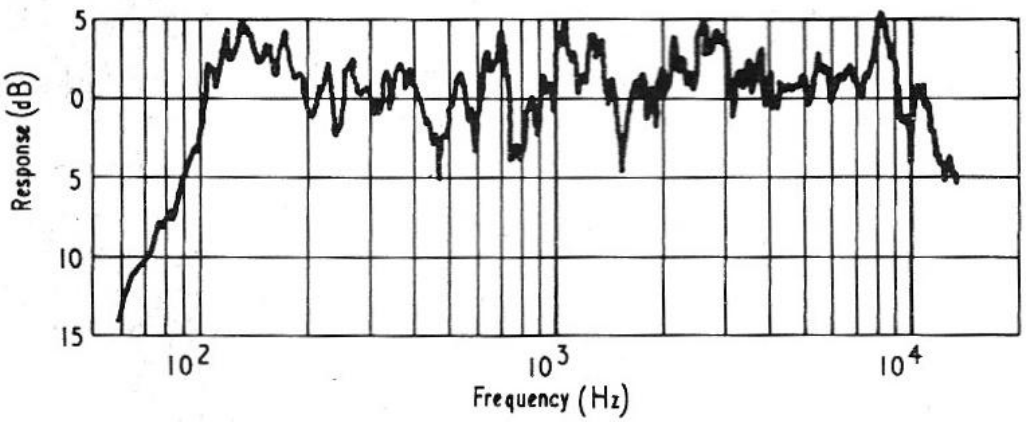
$$\int \sin \omega t \; dt = -1/\omega \, . \cos \omega t$$

and

$$\int \cos \omega t \; dt = 1/\omega \, . \sin \omega t$$

In other words, sine and cosine functions are relatively unaffected by the processes of differentiation and integration. So, analysing a circuit (or solving a differential equation - it comes to the same thing) for sine wave excitation, simplifies the solution to such a degree that the calculus disappears! This is a fantastically important conclusion, because - if only we could assume that our physical and electrical systems were only ever excited by single-frequency forcing functions - we could deduce solutions by simple algebraic manipulations and never have to resort to horrid, differential equations.

Here, the early, nineteenth century, French genius of Jean Baptiste Fourier comes riding to our rescue. Quite wonderfully, it turns out that we can! The Fourier transform demonstrates that any forcing function may be considered to be a mixture of sine waves and cosine waves. Provided the system is linear, we can consider the performance by considering the sum of its responses to single-frequency forcing functions. That is the reason why hi-fi magazines can publish a graph like this one,



which illustrates the amplitude response of the loudspeaker (how loud it sounds), in response to a slowly changing, constant-voltage, sinusoidal signal, “sweeping’ from the sub-bass to the ultra-sonic. Although nobody chooses to listen to sine waves, because we know how the loudspeaker reacts to each frequency, and we know that music (however apparently lush and complicated) is made up of a sum of sine waves, we can predict the performance of the loudspeaker from these curves.

Similarly, by considering only single-frequency forcing functions, the solution differential equation which describes the circuit in Fig. 9, simplifies to one in which (ω) becomes a straightforward scaling factor so that,

$$v(t) = i\,(R + j\omega L + 1/(j\omega C))$$

which brings us to the subject of *transfer functions.*

Transfer functions

Knowing how to calculate impedances enables us to predict their interaction using the AC version of Ohm’s law and thereby simulate the performance of a circuit at a particular frequency. Using a computer, it’s possible to make these calculations many, many times and plot how the circuit functions across a whole range of frequencies. Fig. 10 illustrates three simple, passive circuits analysed in just this way using a simple spreadsheet to calculate and plot the results. In each case, the impedances are applied in simple calculations of Ohm’s law and an expression for ***Vout*** derived in terms of ***Vin***. For all its sophistication, circuit analysis software works in exactly this way.

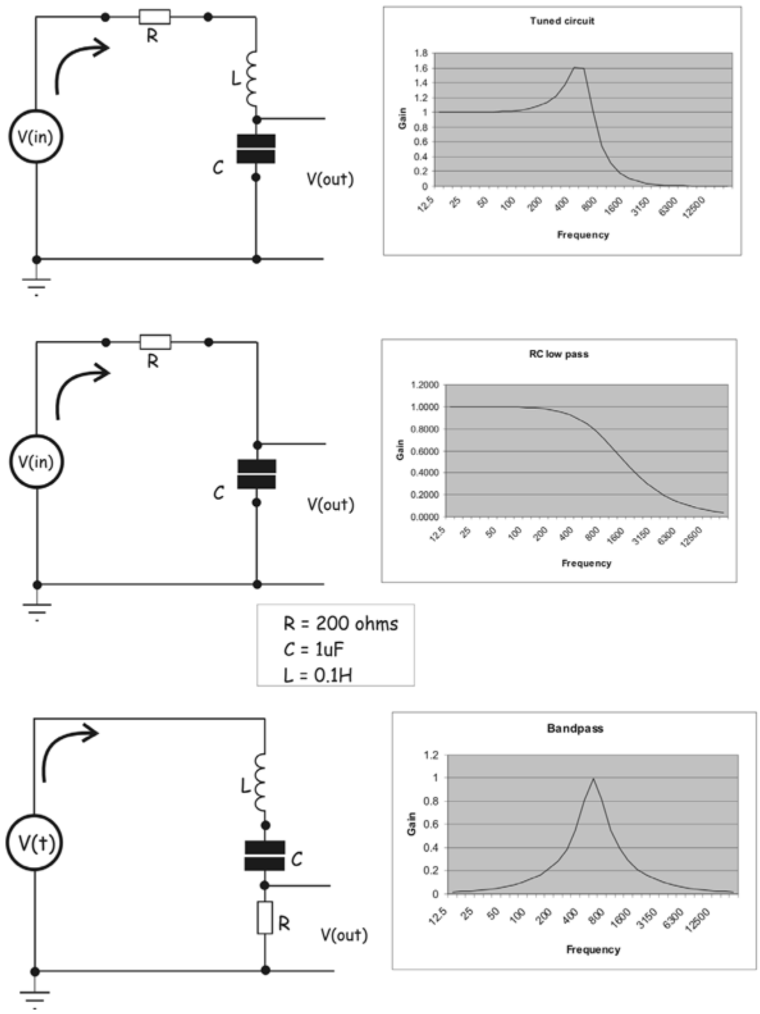
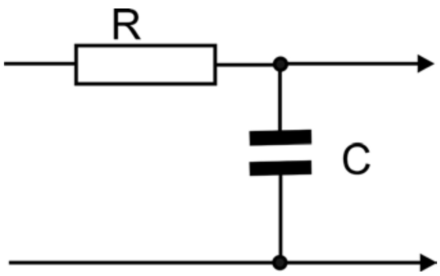


Figure 10 - Three passive circuits analysed using a simple spreadsheet program

However, it is often useful to derive, not concrete, numerical expressions for particular circuits of particular values at particular frequencies, but expressions which relate the output of the circuit to the input as a function of frequency. This is known as a *transfer function*. This is an especially useful technique because, as we shall see in later chapters, more complicated transfer functions may be built-up from the combinations of simple functions, greatly simplifying design procedures.

Taking the simplest of circuits first, let us derive a transfer function for the simple low-pass, RC network like this,



Now, we know,

Vout/Vin = Zb / (Za + Zb)

Impedance ***Za*** is resistive and real and equal to ***R***. Impedance ***Zb*** is *imaginary* and equal to,

$$-\mathbf{j} \cdot 1/\omega C$$

An expression which is not in an especially friendly form. So to render it more neatly, it is common to see it multiplied top and bottom by **j**, to give us,

$1/\mathrm{j}\omega C$ as the expression for the impedance of a capacitor.

Substituting these expressions into the equation for ***Vout/Vin*** we get,

$$Vout/Vin = (1/\mathrm{j}\omega C) / ((1/\mathrm{j}\omega C) + R)$$

Which, if we multiply top and bottom by $\mathrm{j}\omega C$, we derive,

$$Vout/Vin = 1 / (1 + \mathrm{j}\omega CR)$$

At which point we've almost entirely done the job, except that ideally we'd like to remove all traces of resistors and capacitors because we might want to apply this knowledge to other situations. For example, we might want to define a transfer function for a loudspeaker in which the impedances are mechanical. In order to accomplish this, the product of ***RC*** is expressed as a *time-constant*, often annotated as ***T*** [5]. So the final form of transfer function is,

$$H(\mathrm{j}\omega) = 1 / (1 + \mathrm{j}\omega T)$$

where the $H(\mathrm{j}\omega)$ means the expression is a function (here expressed with an **H** rather than an **F**) of complex frequency ($\mathrm{j}\omega$).

Magnitude of the transfer function

Let’s look at this expression. It shows that when ω is zero, the denominator is real and the expression is equal to one; meaning that a circuit with this characteristic passes zero-frequency (or DC) with no attenuation. At the other end of the scale, when frequency is high and ω is a big number, the expression will tend to zero: it will never actually get there, but as frequency increases, less and less signal is *transferred* from input to output. In between these extremes, there exists an important watershed frequency when $T\omega = 1$. At this frequency,

$$H(\mathrm{j}\omega) = 1 / (1 + \mathrm{j}1)$$

Remember that the real part of the complex denominator doesn’t add arithmetically to the imaginary part. Instead we have to calculate the modulus (or magnitude) complex number, which we do by following the rule that the magnitude of complex number is equal to the square root of the square of the real and imaginary parts. So we get,

$$|H(\mathrm{j}\omega)| = 1 / (\sqrt{ [1^2 + 1^2]})$$

$$= 1 / \sqrt{2}$$

$$|H(\mathrm{j}\omega)| = 0.707$$

As we saw before, this is equivalent to 3dB attenuation in voltage terms or ½ power and it is termed the *breakpoint*, *turning* or *turnover* frequency of the network. If we calculate the modulus of the transfer function for a range of frequencies, we can plot a curve which

---

[5] Note that, when an inductor and a resistor are involved, the time-constant is given by, $T = L/R$.

illustrates how the magnitude of the transfer function changes with frequency. This may be applied to any combination of ***R*** and ***C*** possible.

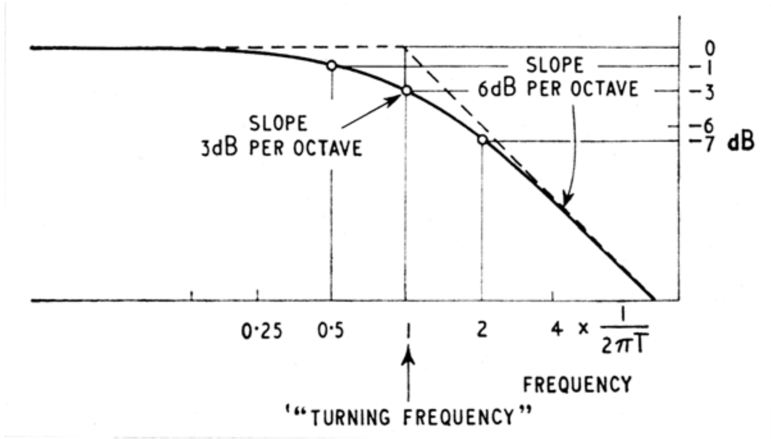


Figure 11 - Magnitude response of a simple, low-pass RC circuit

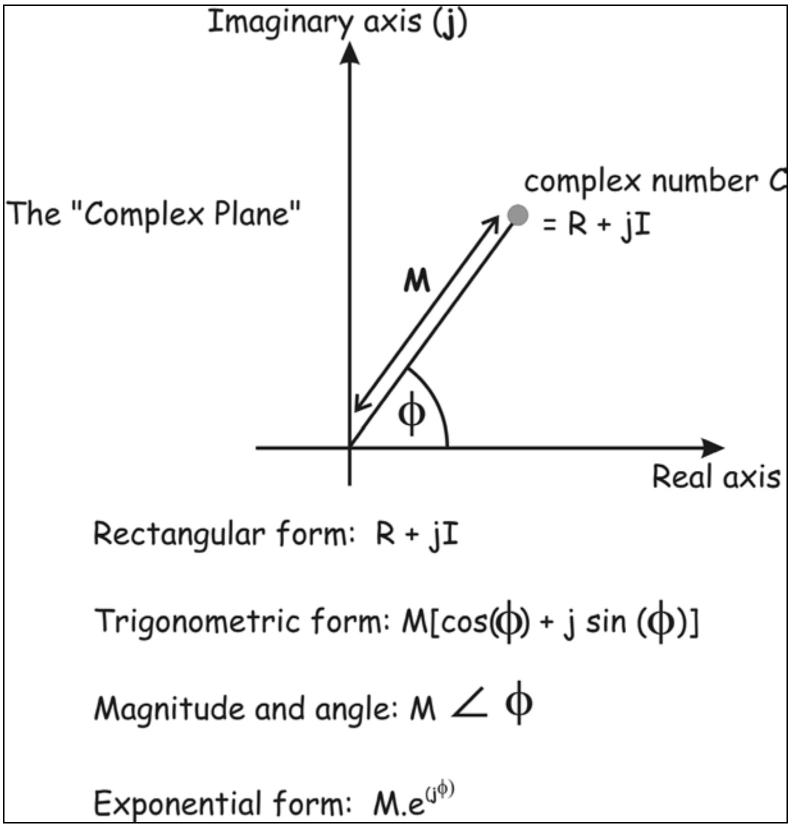


Figure 12 - Key concepts of complex numbers

Phase of the transfer function

Remember the transfer function is a complex function, so it does not simply have a magnitude, it has an *argument* (or *phase*) as well. The function **H(jω)** enables us to predict the frequency dependent phase-shift caused by the low-pass ***RC*** network. The argument of the complex number (***x*** **+ j** ***y***) is defined by,

$$\tan \varphi = y/x$$

In the case of the expression **$H(j\omega) = 1/(1 + j\omega T)$**, we have the complication that the denominator is a complex number, so we need to remove it by multiplying the denominator and numerator by its conjugate. This leaves us with the expression,

$$\mathbf{H(j\omega) = (1 - j\omega T) / [(1 + j\omega T) . (1 - j\omega T)]}$$

$$\mathbf{= (1 - j\omega T) / (1 + \omega^2.T^2)}$$

Which gives us the complex number,

$$\mathbf{[1 / (1 + \omega^2.\mathit{T}^2)] + [j\omega T / (1 + \omega^2.\mathit{T}^2)]}$$

So the phase angle may be calculated as,

$$\mathbf{\tan \varphi = [-\omega \mathit{T} / (1 + \omega^2.T^2)] / [1 / (1 + \omega^2.T^2)]}$$

$$\mathbf{= -\omega \mathit{T}}$$

The values of **φ** have been plotted for various frequencies in Fig. 13 and - once again - this is a general form and may be applied to any RC, low-pass circuit.

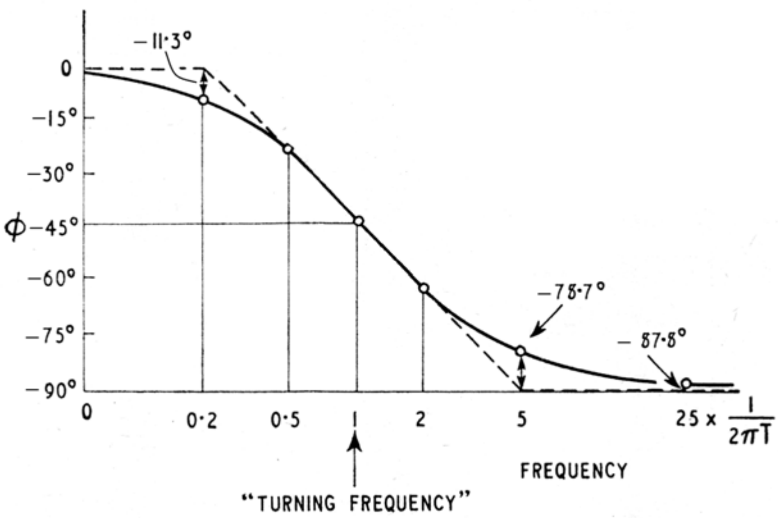
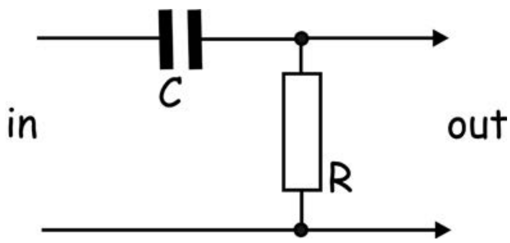


Figure 13 - Phase response of low-pass circuit

Now, let's look at a high-pass, CR network in which the C and R are transposed.



Once again the output will depend on the ratio of impedances ***Za*** and ***Zb*** such that,

$$\mathbf{\mathit{Vout/Vin = Zb / (Za + Zb)}}$$

But here the appropriate substitution will give,

$$\mathbf{\mathit{Vout/Vin = R / ((1/}j\omega \mathit{C) + R)}}$$